

# Long-Run Analysis of the Stochastic Replicator Dynamics in the Presence of Random Jumps

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## Abstract

We further generalize the stochastic version of the replicator dynamics due to Fudenberg and Harris [7]. In particular, we add a random jump term to the payoff function to simulate anomalous events and their effects on the fitness. Assuming a  $2 \times 2$  game and using a particular characteristic of the jump functions we are able to estimate the ergodic measure for all games. Lastly, working with results and methods developed by Imhoff [11], we prove some stability theorems for an arbitrary  $n \times n$  game.

## 1 Introduction

Consider a two-player symmetric game, where  $a_{ij}$  is the payoff to a player using strategy  $S_i$  against an opponent employing strategy  $S_j$ , and define  $A = (a_{ij})$ . Furthermore, define  $\Delta_n = \left\{ \mathbf{y} \in \mathbb{R}^n : y_i > 0 \text{ for all } i \text{ and } \sum y_i = 1 \right\}$  and  $\bar{\Delta}_n$  as the closure. Within a population we assume that every individual is programmed to play a pure strategy. Let  $r_i(t)$  be the size of the subpopulation that plays strategy  $S_i$  at time  $t$ , which we denote as the  $i^{\text{th}}$  subpopulation. Furthermore, define  $\mathbf{r}(t) := (r_1(t), \dots, r_n(t))^T$ ,  $R(t) := \sum_i r_i(t)$  and  $\mathbf{s}(t) := (s_1(t), \dots, s_n(t))^T$  where  $s_i(t) := r_i(t)/R(t)$  (the  $i^{\text{th}}$  subpopulation fraction of the population). When a player in the  $i^{\text{th}}$  subpopulation is randomly matched with another player from the entire population,  $(\mathbf{A}\mathbf{s}(t))_i$  is the average payoff for this individual, and we take this to be the fitness of the player. We assume growth is proportional to fitness:

$$\dot{r}_i(t) = r_i(t)(\mathbf{A}\mathbf{s}(t))_i,$$

and hence

$$\dot{s}_i(t) = s_i(t) \left( (\mathbf{A}\mathbf{s}(t))_i - \mathbf{s}(t)^T \mathbf{A}\mathbf{s}(t) \right).$$

This is the deterministic version of the replicator dynamic.

Fudenberg and Harris [7] make this model stochastic by assuming

$$dr_i(t) = r_i(t) \left( (\mathbf{A}\mathbf{s}(t))_i dt + \sigma_i dW_i(t) \right),$$

for  $\sigma_i \in \mathbb{R}_+$  and  $W_i(t)$  a pairwise independent standard Wiener processes. Itô's lemma then yields

$$ds_i(t) = \sum_{j \neq i} s_i(t)s_j(t) \left[ \left( (\mathbf{A}\mathbf{s}(t))_i - (\mathbf{A}\mathbf{s}(t))_j \right) dt + \left( \sigma_j^2 s_j(t) - \sigma_i^2 s_i(t) \right) dt + \left( \sigma_i dW_i(t) - \sigma_j dW_j(t) \right) \right]. \quad (1)$$

The idea behind this model is that randomness comes from the fitness of each subpopulation, and hence through fluctuations in the payoffs. The authors then take a 2 strategy game, which makes  $s_2(t) = 1 - s_1(t)$ , and obtain the more explicit model

$$\begin{aligned} ds_1(t) = & s_1(t) \left( 1 - s_1(t) \right) \left[ a_{12} - a_{22} + \sigma_2^2 + \left\{ a_{11} - a_{21} - \sigma_1^2 + a_{22} - a_{12} - \sigma_2^2 \right\} s_1(t) \right] dt \\ & + \sigma s_1(t) \left( 1 - s_1(t) \right) dW(t). \end{aligned} \quad (2)$$

This takes us to a one-dimensional analysis in which there are many results. The authors are then able to determine the ergodic measure for all games.

The Fudenberg and Harris [7] model simulates everyday noise very well, however, it fails to capture the impacts on the fitness of anomalies. There are many examples of these events, such as earthquakes, tsunamis, volcanic explosions, floods, an exceptional amount of toxic bonds in the stock market, etc. These are all one-time events that have a residual effect on the fitnesses of the subpopulations. To account for these effects we add a Poisson jump-term to the Fudenberg and Harris model and derive a jump stochastic differential equation version of the replicator dynamic. We then approximate the ergodic measure for a  $2 \times 2$  game and a specific jump function. Lastly, following the work of Imhof [11], we derive some stability results in a general  $n$  subpopulation ( $n \times n$  game).

## 2 The Jump Model

Take  $X$  to be an  $\mathbb{R}$ -valued Lévy process on  $(\Omega, \mathcal{F}, P)$  with the filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ , where  $\mathcal{F}_0$  contains all of the null sets of  $\mathcal{F}$ . For  $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$  and a fixed  $\omega \in \Omega$ , define  $\Delta X_\omega(s) := X_\omega(s) - X_\omega(s-)$  and  $N_\omega(t, B) := \#\{0 \leq s \leq t : \Delta X_\omega(s) \in B\}$ . This random counting measure is known as the **Poisson measure** since, fixing  $B$ , the map  $t \rightarrow N_\omega(t, B)$  is a Poisson process with intensity  $t\nu(B) := E[N_\omega(t, B)]$ . We call  $\nu(\cdot)$  the **intensity measure** and it is well known that it is a Lévy measure, hence, it is a Borel Measure with  $\int_{\mathbb{R} \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty$ .

The measure  $\tilde{N}_\omega(dt, dx) := N_\omega(dt, dx) - dt\nu(dx)$  is called the **compensated Poisson measure** (where  $dt$  is the Lebesgue measure). For  $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$  where  $0 \notin \overline{B}$ , one can show  $\tilde{N}_\omega(t, B)$  is a martingale and  $E[\tilde{N}_\omega(t, B)] = 0$ , (which stems from  $\nu(B) < \infty$ ). See Sato [18], Bertoin [3], or Applebaum [2] for further information.

Assuming  $\nu(\mathbb{R}) < \infty$ , we have  $E[\tilde{N}_\omega(t, \mathbb{R})] = 0$  and that there are only finitely many jumps in a finite interval of time. It is because of these characteristics that we add the term  $\int_0^t \int_{\mathbb{R}} h_i(x) \tilde{N}(dt, dx)$  to the fitness of each respective  $i^{th}$  subpopulation.

**Definition 2.1.** We call  $h_i(x)$  the **jump-affect** of the  $i^{th}$  subpopulation.

Thus our growth model is

$$dr_i(t) = r_i(t-) \left( (As(t-))_i dt + \sigma_i dW_i(t) + \int_{\mathbb{R}} h_i(x) \tilde{N}(dt, dx) \right). \quad (3)$$

We use the remark below to motivate the next assumption.

**Remark 2.1.** If we assume that  $\inf_{x \in \mathbb{R}} \{h_i(x)\} > -1$  then  $r_i(t)$  can be rewritten as  $\exp(Y_i(t))$  where  $dY_i(t) = \left( (As(t-))_i - \frac{\sigma_i^2}{2} \right) dt + \sigma_i dW_i(t) + \int_{\mathbb{R}} \log[1 + h_i(x)] \tilde{N}(dt, dx) + \int_{\mathbb{R}} \left( \log[1 + h_i(x)] - h_i(x) \right) \nu(dx) dt$ . This follows from Itô's lemma.

**Assumption 2.1.** We assume that  $\nu(\mathbb{R}) < \infty$ . Moreover, for all  $i$ :

- a.  $h_i(x)$  is bounded;
- b.  $\inf_{x \in \mathbb{R}} \{h_i(x)\} > -1$ ;
- c.  $h_i(x)$  is continuously differentiable.

**Note 2.1.** *Although the assumption above was presented in the context of technical issues, (guaranteeing that we have exponential growth), it make sense in population dynamics. If an anomaly has an incredibly detrimental impact on the populace then the entire dynamics has completely changed. An example of this type of impact is the polar bear and strategies for finding food coupled with the polar ice caps melting.*

We will assume a two subpopulation model for the rest of this section and the following section as well. In the latter sections we show conditions for certain stabilities for a general  $n$  subpopulation. Applying Itô's lemma ([8] Theorem 2 Chapter 2 §9, [2]) to  $s_1(t)$  yields

$$\begin{aligned}
ds_1(t) = & \left[ s_1(t-)s_2(t-) \left( (As(t-))_1 - (As(t-))_2 + s_2(t-)\sigma_2^2 - s_1(t-)\sigma_1^2 \right) \right. \\
& + \int_{\mathbb{R}} \left( \frac{s_1(t-) + s_1(t-)h_1(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-)}{s_1(t-) + s_2(t-)} \right. \\
& \left. \left. - [s_2(t-)h_1(x)s_1(t-) - s_2(t-)h_2(x)s_1(t-)] \right) \nu(dx) \right] dt \\
& + s_1(t-)s_2(t-) (\sigma_1 dW_1(t) - \sigma_2 dW_2(t)) \\
& + \int_{\mathbb{R}} \left( \frac{s_1(t-) + s_1(t-)h_1(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-)}{s_1(t-) + s_2(t-)} \right) \tilde{N}(dt, dx).
\end{aligned} \tag{4}$$

Solving for  $ds_2(t)$  gives us a similar equality. This particular version of Itô's lemma can be found in Applebaum (Theorem 4.4.7 [2]) or Gihman and Skorohod ([8] Part II Chapter 2 §6).

**Proposition 2.1.** *For all finite  $t \geq 0$  and  $\mathbf{y} \in \Delta_n$ , we have  $P_{\mathbf{y}}(\mathbf{s}(t) \in \Delta_n) = 1$ .*

*Proof.* The following proof is done for the  $n = 2$  case, however, the methods used can be applied for general  $n$ . We will first show that for  $t \geq 0$ , we have  $s_1(t) + s_2(t) = 1$  a.s. Define the map  $(s_1(t), s_2(t)) \rightarrow \sum_{i=1}^2 s_i(t) := G(t)$ , and set  $G(0) = 1$ . Itô's lemma gives us that

$$\begin{aligned}
dG(t) = & \left\{ \left[ s_1(t-)s_2(t-) \left( (As(t-))_1 - (As(t-))_2 + s_2(t-)\sigma_2^2 - s_1(t-)\sigma_1^2 \right) \right. \right. \\
& + \int_{\mathbb{R}} \left( \frac{s_1(t-) + s_1(t-)h_1(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-)}{s_1(t-) + s_2(t-)} \right. \\
& \left. \left. - [s_2(t-)h_1(x)s_1(t-) - s_2(t-)h_2(x)s_1(t-)] \right) \nu(dx) \right] dt + s_1(t-)s_2(t-) (\sigma_1 dW_1(t) - \sigma_2 dW_2(t)) \right\} \\
& + \left\{ \left[ s_2(t-)s_1(t-) \left( (As(t-))_2 - (As(t-))_1 + s_1(t-)\sigma_1^2 - s_2(t-)\sigma_2^2 \right) \right. \right. \\
& + \int_{\mathbb{R}} \left( \frac{s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_2(t-)}{s_1(t-) + s_2(t-)} \right. \\
& \left. \left. - [s_1(t-)h_2(x)s_2(t-) - s_1(t-)h_1(x)s_2(t-)] \right) \nu(dx) \right] dt + s_1(t-)s_2(t-) (\sigma_2 dW_2(t) - \sigma_1 dW_1(t)) \right\}
\end{aligned}$$

$$\begin{aligned}
& - \left[ s_1(t-)h_2(x)s_2(t-) - s_1(t-)h_1(x)s_2(t-) \right] \nu(dx) \Big] dt + s_2(t-)s_1(t-) \left( \sigma_2 dW_2(t) - \sigma_1 dW_1(t) \right) \Big\} \\
& - \int_{\mathbb{R}} \left( \left[ s_1(t-) + \frac{s_1(t-) + s_1(t-)h_1(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-)}{s_1(t-) + s_2(t-)} \right] \right. \\
& + \left[ s_2(t-) + \frac{s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_2(t-)}{s_1(t-) + s_2(t-)} \right] \\
& \left. - (s_1(t-) + s_2(t-)) \right) \nu(dx) dt \\
& + \int_{\mathbb{R}} \left( \left[ s_1(t-) + \frac{s_1(t-) + s_1(t-)h_1(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-)}{s_1(t-) + s_2(t-)} \right] \right. \\
& + \left[ s_2(t-) + \frac{s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_2(t-)}{s_1(t-) + s_2(t-)} \right] \\
& \left. - (s_1(t-) + s_2(t-)) \right) \tilde{N}(dt, dx) \\
& = \int_{\mathbb{R}} \left[ \frac{s_1(t-) + s_1(t-)h_1(x) + s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-) + s_2(t-)}{s_1(t-) + s_2(t-)} \right] \nu(dx) dt \\
& - \int_{\mathbb{R}} \left[ \frac{s_1(t-) + s_1(t-)h_1(x) + s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-) + s_2(t-)}{s_1(t-) + s_2(t-)} \right] \nu(dx) dt \\
& = \int_{\mathbb{R}} \left[ \frac{s_1(t-) + s_1(t-)h_1(x) + s_2(t-) + s_2(t-)h_2(x)}{(s_1(t-) + s_1(t-)h_1(x)) + (s_2(t-) + s_2(t-)h_2(x))} - \frac{s_1(t-) + s_2(t-)}{s_1(t-) + s_2(t-)} \right] \tilde{N}(dt, dx) \\
& = \int_{\mathbb{R}} (1 - 1) \tilde{N}(dt, dx) = 0 \quad a.s.
\end{aligned}$$

Finally, we show that the stochastic replicator dynamics does not hit or jump over the boundary in finite time. Define  $\Psi(\mathbf{y}) = \log(y_1/y_2)$ ,  $\tau$  as the first time this dynamic leaves the simplex (i.e., such that  $s_i(\tau) \leq 0$  for some  $i$ ), and  $Z(t) := \Psi(\mathbf{s}(t))$  for  $t < \tau$ . Itô's lemma yields

$$\begin{aligned}
dZ(t) &= \left[ (1, -1)^T A \Psi^{-1}(Z(t-)) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2) + \int_{\mathbb{R}} \left( h_2(x) - h_1(x) + \log \left( \frac{1 + h_1(x)}{1 + h_2(x)} \right) \right) \nu(dx) \right] dt \\
&+ \sigma_1 dW_1(t) - \sigma_2 dW_2(t) + \int_{\mathbb{R}} \log \left( \frac{1 + h_1(x)}{1 + h_2(x)} \right) \tilde{N}(dx, dt)
\end{aligned}$$

where

$$\begin{aligned}
s_1(t) &= \frac{s_1(t)}{s_1(t) + s_2(t)} = \frac{s_1(t)/s_2(t)}{s_1(t)/s_2(t) + 1} = \frac{e^{Z(t)}}{e^{Z(t)} + 1}, \\
s_2(t) &= \frac{1}{e^{Z(t)} + 1},
\end{aligned}$$

and

$$\Psi^{-1}(y) = \frac{1}{e^y + 1} (e^y, 1)^T.$$

Take  $\mathfrak{L}$  as the infinitesimal generator for  $Z(t)$ . To apply Theorem 2.1 in Meyn and Tweedie [17], we need to show that there exists a non-negative  $\varphi \in C^2(\mathbb{R})$  such that  $\lim_{|y| \rightarrow \infty} \varphi(y) = \infty$ , and a positive constant  $\lambda$  where  $\mathfrak{L}\varphi \leq \lambda\varphi$ , in order to conclude that  $P(\tau = \infty)$ . By Theorem 2 (Part II Chapter 2 §9) in Gihman and Skorohod [8] (or Chapter 6 §7 [2]), the generator of  $Z(t)$  has the form

$$\begin{aligned}
\mathfrak{L}\varphi(y) &= a^*(y) \frac{d\varphi}{dy}(y) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \frac{d^2\varphi}{dy^2}(y) \\
&+ \int_{\mathbb{R}} \left[ \varphi \left( y + \log \left( \frac{1 + h_1(x)}{1 + h_2(x)} \right) \right) - \varphi(y) \right] \nu(dx),
\end{aligned}$$

where

$$a^*(y) = (1, -1)^T A \Psi^{-1}(y) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2) + \int_{\mathbb{R}} (h_2(x) - h_1(x)) \nu(dx).$$

Note that  $|a(y)|$  is bounded by a constant, say  $K$ . Now take  $\varphi(y) = 1 + y^2$ . Then

$$\begin{aligned} \mathfrak{L}\varphi(y) &= 2a(y)y + (\sigma_1^2 + \sigma_2^2) + \int_{\mathbb{R}} \left[ \left( y + \log \left( \frac{1+h_1(x)}{1+h_2(x)} \right) \right)^2 - y^2 \right] \nu(dx) \\ &= 2 \left( a(y) + \int_{\mathbb{R}} \log \left( \frac{1+h_1(x)}{1+h_2(x)} \right) \nu(dx) \right) y + (\sigma_2^2 - \sigma_1^2) + \int_{\mathbb{R}} \log^2 \left( \frac{1+h_1(x)}{1+h_2(x)} \right) \nu(dx) \\ &\leq 2 \left( K + \int_{\mathbb{R}} \log \left( \frac{1+h_1(x)}{1+h_2(x)} \right) \nu(dx) \right) y + (\sigma_1^2 + \sigma_2^2) + \int_{\mathbb{R}} \log^2 \left( \frac{1+h_1(x)}{1+h_2(x)} \right) \nu(dx) := \phi(y). \end{aligned}$$

Defining  $\lambda = \max_{y \in [-1, 1]} |\phi(y)|$  and noting that  $\phi$  is linear, yields  $\mathfrak{L}\varphi \leq \lambda\varphi$ . Therefore, by Theorem 2.1 in Meyn and Tweedie [17],  $P(\tau = \infty) = 1$ , which finishes the proof.  $\square$

By the proposition we have the equality  $s_2(t) = 1 - s_1(t)$ . Hence, we may just focus on the dynamics of

$$\begin{aligned} ds_1(t) &= s_1(t-) \left( 1 - s_1(t-) \right) \left[ a_{12} - a_{22} + \sigma_2^2 + \int_{\mathbb{R}} \left( \frac{h_1(x) - h_2(x)}{s_1(t-) [h_1(x) - h_2(x)] + 1 + h_2(x)} + h_2(x) - h_1(x) \right) \nu(dx) \right. \\ &\quad \left. + \left( a_{11} - a_{21} - \sigma_1^2 + a_{22} - a_{12} - \sigma_2^2 \right) s_1(t-) \right] dt \\ &\quad + \sigma s_1(t-) \left( 1 - s_1(t-) \right) dW(t) \\ &\quad + \int_{\mathbb{R}} \frac{s_1(t-) (1 - s_1(t-)) [h_1(x) - h_2(x)]}{s_1(t-) [h_1(x) - h_2(x)] + 1 + h_2(x)} \tilde{N}(dt, dx), \end{aligned} \tag{5}$$

where  $\sigma := \sqrt{\sigma_1^2 + \sigma_2^2}$  and  $W(t) := \frac{\sigma_1 W_1(t) - \sigma_2 W_2(t)}{\sigma}$ .

**Remark 2.2.** Defining  $H(s(t-)) := \int_{\mathbb{R}} \left( \frac{h_1(x) - h_2(x)}{s_1(t-) [h_1(x) - h_2(x)] + 1 + h_2(x)} + h_2(x) - h_1(x) \right) \nu(dx)$ , the drift coefficient can be written as

$$\begin{aligned} &s_1(t-) \left( 1 - s_1(t-) \right) \left[ \left( a_{12} - a_{22} + \sigma_2^2 + H(s(t-)) \right) \right. \\ &\quad \left. + \left( \left( a_{11} - a_{21} - \sigma_1^2 + H(s(t-)) \right) + \left( a_{22} - a_{12} - \sigma_2^2 - H(s(t-)) \right) \right) s_1(t-) \right]. \end{aligned} \tag{6}$$

Moreover, if  $h_1(x) = h_2(x)$  for every  $x \in \mathbb{R}$  then the jump terms disappear. Recall that for  $\tilde{A} := \begin{pmatrix} a_{11} + c & a_{12} \\ a_{21} + c & a_{22} \end{pmatrix}$  or  $\tilde{A} := \begin{pmatrix} a_{11} & a_{12} + c \\ a_{21} & a_{22} + c \end{pmatrix}$ , we have the equality  $(\tilde{A}p)_i - p \cdot \tilde{A}p = (Ap)_i - p \cdot Ap$  for  $i = 1, 2$  [10], which is what we would expect. Furthermore, if an anomaly were to affect the environment in such a way that the subpopulations are equally effected, we would not expect unequal change in their fitnesses. For an example of such an event consider the scorpion population in the American southwest and how each subpopulation is subjected to flooding. The amount of food is impacted equally across each subpopulation as well as the chances of individuals drowning.

### 3 An Approximation of the Ergodic Measure

In this section we determine the stability for a  $2 \times 2$  game for a certain class of jump-affect functions. In particular, we give conditions that estimate the long-run behavior of our process. Since  $s_2(t) = 1 - s_1(t)$ , we simplify the analysis by only considering the dynamics of  $s_1(t)$  with respect to Equation (5) (Theorem 2 Part II Chapter 2 §9 [8]). Notice for

$$\tilde{\alpha}(y) =: y(1-y) \left[ \left( a_{12} - a_{22} + \sigma_2^2 + \int_{\mathbb{R}} (h_2(x) - h_1(x)) \nu(dx) \right) + \left( (a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2) \right) y \right],$$

$$\tilde{\beta}(y) := \frac{\sigma^2}{2} y^2 (1-y)^2,$$

and

$$y + \gamma(y, x) := y + \frac{y(1-y)[h_1(x) - h_2(x)]}{y[h_1(x) - h_2(x)] + 1 + h_2(x)} = \frac{y[1 + h_1(x)]}{y[h_1(x) - h_2(x)] + 1 + h_2(x)},$$

and taking  $L$  as the infinitesimal generator of  $s_1(t)$  given by Equation (5), we have

$$Lf(\cdot) = \tilde{\alpha}(\cdot)f'(\cdot) + \tilde{\beta}(\cdot)f''(\cdot) + \int_{\mathbb{R}} \left( f(\cdot + \gamma(\cdot, x)) - f(\cdot) \right) \nu(dx).$$

Now, for  $0 < y_1 < y_2 < 1$ , define  $\tau_{y_1 y_2}(y_0) = \inf_{t \geq 0} \left\{ s_1(t) \notin (y_1, y_2) \mid s_1(0) = y_0 \right\}$ ,  $\pi_{y_2; y_1}(y_0) = P\left(s_1(\tau_{y_1 y_2}(y_0)) \geq y_2\right)$ , and  $\pi_{y_1; y_2}(y_0) = P\left(s_1(\tau_{y_1 y_2}(y_0)) \leq y_1\right)$ .

Consider an integro-differential equation on the unit interval of the form

$$Lu(y) = \tilde{\alpha}(y)u'(y) + \tilde{\beta}(y)u''(y) + \int_{\mathbb{R}} \left[ u(y + \gamma(y, x)) - u(y) \right] \nu(dx) = 0 \quad \text{for } y \in (y_1, y_2), \quad (7)$$

with the conditions  $u(y) = 0$  for  $y \in [0, y_1]$ , and  $u(y) = 1$  for  $y \in [y_2, 1]$ . The papers of Henry Tuckwell [22] and Mario Abundo [1] tell us that solving this integro-differential equation will give us  $\pi_{y_2; y_1}(y_0)$ , (interchanging the initial conditions will give  $\pi_{y_1; y_2}(y_0)$ ). However, in order to apply these theorems we need to verify the condition  $E[\tau_{y_1 y_2}^n(y_0)] < \infty$  for every  $n \in \mathbb{Z}_+$ . This property is shown in Theorem 4.1.

**Remark 3.1.** *We should note here that the result in Tuckwell [22] is for a jump-diffusion with a Poisson measure and not the compensated Poisson measure. However, Tuckwell's proof is based on a result in Gihman and Skorohod [8], in which the authors give an equality for the transition probability for a jump-diffusion with a compensated Poisson measure (Part II Chapter 2 §9). Adjusting the first order coefficient by adding the integral with respect to the Lévy measure and proceeding similarly will give the equivalent conclusions.*

Solving this integro-differential equation is a very difficult task and so we will construct a way to approximate the solution. First we assume that  $h_1(x) = h_2(x) + \epsilon$ , for a small  $\epsilon \in \mathbb{R}$ . This assumption tells us that one subpopulation fairs a little better than the other. Next we will turn the difference in the integral into a Taylor series, using  $\epsilon$  as the variable, grouping the higher order terms into an error term. Note that we will use the function  $f$ , instead of  $u$ , to find the solution, since normalizing  $f$  and considering the initial conditions will determine  $u$ .

Before we begin we need the function  $f$  to be in an appropriate space. Given  $0 < y_1 < y_2 < 1$ , define  $\tilde{y}_1 = \min_{x \in \mathbb{R}, y \in (y_1, y_2)} \{y_1, y + \gamma(y, x)\}$  and  $\tilde{y}_2 = \max_{x \in \mathbb{R}, y \in (y_1, y_2)} \{y_2, y + \gamma(y, x)\}$ . Note that  $0 < \tilde{y}_1 < \tilde{y}_2 < 1$  since  $y + \gamma(y, x) = 0$  if and only if  $y = 0$  and  $y + \gamma(y, x) = 1$  if and only if  $y = 1$ . Define  $C_{y_1 y_2}$  as the space of bounded continuous functions that map  $[\tilde{y}_1, \tilde{y}_2]$  to  $\mathbb{R}$ , and take  $f \in \left\{ g \in C_{y_1 y_2} : g' \in C_{y_1 y_2} \text{ and } g'' \in C_{y_1 y_2} \right\}$ .

By our assumption on the jump-affect functions, the integral difference term is

$$\int_{\mathbb{R}} \left[ f\left(\frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)}\right) - f(y) \right] \nu(dx) := F(\epsilon).$$

Then, by our assumption on  $f$ ,  $F(0) = 0$ ,

$$\begin{aligned} F'(0) &= \int_{\mathbb{R}} f' \left( \frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left( \frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)} \right)' \nu(dx) \Big|_{\epsilon=0} \\ &= \int_{\mathbb{R}} f' \left( \frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left( \frac{y(1-y)[1 + h_2(x)]}{(\epsilon y + 1 + h_2(x))^2} \right) \nu(dx) \Big|_{\epsilon=0} \\ &= y(1-y) f'(y) \int_{\mathbb{R}} \frac{1}{1 + h_2(x)} \nu(dx) = C_1 y(1-y) f'(y), \end{aligned}$$

and

$$\begin{aligned} F''(0) &= \int_{\mathbb{R}} \left\{ f'' \left( \frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left( \frac{y(1-y)[1 + h_2(x)]}{(\epsilon y + 1 + h_2(x))^2} \right)^2 \right. \\ &\quad \left. + f' \left( \frac{y[1 + h_2(x) + \epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left( \frac{-2y^2(1-y)[1 + h_2(x)]}{(\epsilon y + 1 + h_2(x))^3} \right) \right\} \nu(dx) \Big|_{\epsilon=0} \\ &= -2y^2(1-y) f'(y) \int_{\mathbb{R}} \frac{1}{[1 + h_2(x)]^2} \nu(dx) + y^2(1-y)^2 f''(y) \int_{\mathbb{R}} \frac{1}{[1 + h_2(x)]^2} \nu(dx) \\ &= -2C_2 y^2(1-y) f'(y) + C_2 y^2(1-y)^2 f''(y), \end{aligned}$$

where  $C_1 := \int_{\mathbb{R}} \frac{1}{1 + h_2(x)} \nu(dx)$  and  $C_2 := \int_{\mathbb{R}} \frac{1}{[1 + h_2(x)]^2} \nu(dx)$ . Thus

$$F(\epsilon) = C_1 y(1-y) f'(y) \epsilon + \left( -2C_2 y^2(1-y) f'(y) + C_2 y^2(1-y)^2 f''(y) \right) \epsilon^2/2 + O(\epsilon^3).$$

Excluding the error term, Equation (7) becomes the ordinary differential equation

$$\tilde{\alpha}_\epsilon(y) f'(y) + \tilde{\beta}_\epsilon(y) f''(y) = 0,$$

where

$$\tilde{\alpha}_\epsilon(y) := y(1-y) \left[ \left( a_{12} - a_{22} + \sigma_2^2 + \epsilon(C_1 - \nu(\mathbb{R})) \right) + \left\{ (a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2) - \epsilon^2 C_2 \right\} y \right]$$

and

$$\tilde{\beta}_\epsilon(y) := \left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right) y^2(1-y)^2.$$

**Remark 3.2.** Without the quadratic term in our approximation we would have  $\tilde{\beta}_\epsilon(y) = \left( \frac{\sigma^2}{2} \right) y^2(1-y)^2$ , and if  $\sigma_1 = \sigma_2 = 0$  then  $\tilde{\beta}_\epsilon(y) \equiv 0$ . Hence, the quadratic term is included to insure that  $\tilde{\beta}_\epsilon(y) > 0$  for all  $y \in (0, 1)$ .

**Remark 3.3.** Since we are approximating the exiting time of our stochastic replicator equation, we will define the “new” process as  $\hat{\mathbf{s}}(t) = (\hat{s}_1(t), \hat{s}_2(t))$ .

We are now ready to state the main theorem for this section. The statement of the theorem is given from the perspective of the dynamics of  $\hat{s}_1(t)$ , however, since  $\hat{s}_2(t) = 1 - \hat{s}_1(t)$ , understanding how  $\hat{s}_1(t)$  evolves tells us how  $\hat{s}_2(t)$  evolves as well.

**Theorem 3.1.** Take  $\hat{\mathbf{s}}(t) = (\hat{s}_1(t), \hat{s}_2(t))$  in the remark above,  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ ,  $\sigma_i^2$  the variance of the  $i^{\text{th}}$  subpopulation,  $C_1$  and  $C_2$  defined above, and  $\mathbf{y}_0 = (y_0, y'_0) \in \Delta_2$ .

- (i) If  $a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2$  and  $a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$ , then  $P_{\mathbf{y}_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = 1$ .
- (ii) If  $a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2$  and  $a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$ , then  $P_{\mathbf{y}_0} \left( \limsup_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = P_{\mathbf{y}_0} \left( \liminf_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = 1$ .
- (iii) If  $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2$  and  $a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$ , then  $P_{\mathbf{y}_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = 1$ .
- (iv) If  $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2$  and  $a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$ , then  $P_{\mathbf{y}_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = \frac{f(1) - f(y_0)}{f(1) - f(0)}$  and  $P_{\mathbf{y}_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = \frac{f(y_0) - f(0)}{f(1) - f(0)}$ .

*Proof.* We will prove this theorem by proving four lemmas, where the lemmas correspond to each particular case of inequalities.  $\square$

Before we prove each lemma we give some intuition behind the inequalities in the theorem above, and use these details as a basis of our proofs. Using an integrating factor, we see that

$$f'(y) = k_1 \exp \left\{ - \int_{y_0}^y \frac{\tilde{\alpha}_\epsilon(z)}{\tilde{\beta}_\epsilon(z)} dz \right\}$$

for some constant  $k_1$  and  $y_0 \in (y_1, y_2)$ . We now simplify the integral in the exponent. Define  $N_\epsilon = a_{12} - a_{22} + \sigma_2^2 + \epsilon(C_1 - \nu(\mathbb{R}))$  and  $M_\epsilon = (a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2) - \epsilon^2 C_2$ . So

$$\begin{aligned} - \int_{y_0}^y \frac{\tilde{\alpha}_\epsilon(z)}{\tilde{\beta}_\epsilon(z)} dz &= - \left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} \int_{y_0}^y \frac{N_\epsilon + M_\epsilon z}{z(1-z)} dz \\ &= - \left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} \int_{y_0}^y \frac{N_\epsilon + M_\epsilon}{(1-z)} + \frac{N_\epsilon}{z} dz \\ &= \log \left( \left( y/y_0 \right)^{-\left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} N_\epsilon} \left( (1-y)/(1-y_0) \right)^{\left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} (N_\epsilon + M_\epsilon)} \right). \end{aligned} \tag{8}$$

For some constant  $k_2$  we have

$$\begin{aligned} f(y) &= k_1 \int_{y_0}^y \left( z/y_0 \right)^{-\left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} N_\epsilon} \left( (1-z)/(1-y_0) \right)^{\left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} (N_\epsilon + M_\epsilon)} dz - k_2 \\ &:= k_1 \int_{y_0}^y \left( z/y_0 \right)^{-\hat{N}} \left( (1-z)/(1-y_0) \right)^{\hat{M}} dz - k_2 \\ &= \tilde{k}_1 y^{1-\hat{N}} {}_2F_1 \left( -\hat{M}, 1 - \hat{N}, 2 - \hat{N}, y \right) - \tilde{k}_2 \end{aligned}$$

where  $\hat{M} := \left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} (N_\epsilon + M_\epsilon)$ ,  $\hat{N} := \left( \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2 \right)^{-1} N_\epsilon$  with the assumption  $\hat{N} \neq 1$ ,  ${}_2F_1$  is the hypergeometric function,  $\tilde{k}_1 := \frac{k_1(y_0)^{\hat{N}}}{(1-y_0)^{\hat{M}}(1-\hat{N})}$ , and  $\tilde{k}_2 := -k_1 y_0^{1-\hat{N}} {}_2F_1 \left( -\hat{M}, 1 - \hat{N}, 2 - \hat{N}, y_0 \right) - k_2$ .



Defining  $u(y) = \frac{f(y) - f(y_1)}{f(y_2) - f(y_1)}$ ,  $f(y) = f(y_1)$  for  $y \in [0, y_1]$ , and  $f(y) = f(y_2)$  for  $y \in [y_2, 1]$ , we see that  $\tilde{\alpha}_\epsilon(y)u'(y) + \tilde{\beta}(y)u''(y) = 0$  for  $y \in (y_1, y_2)$ ,  $u(y) = 0$  for  $y \in [0, y_1]$ , and  $u(y) = 1$  for  $y \in [y_2, 1]$ .

Note that

$$\begin{aligned} -\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon > -1 &\iff a_{22} - a_{12} - \sigma_2^2 - \epsilon(C_1 - \nu(\mathbb{R})) > -\sigma^2/2 - \frac{\epsilon^2}{2}C_2 \\ &\iff a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2 \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} (N_\epsilon + M_\epsilon) > -1 &\iff a_{11} - a_{21} - \sigma_1^2 + \epsilon(C_1 - \nu(\mathbb{R})) - \epsilon^2 C_2 > -\sigma^2/2 - \frac{\epsilon^2}{2}C_2 \\ &\iff a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2. \end{aligned}$$

We also conclude that

$$-\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon < -1 \iff a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$$

and

$$\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} (N_\epsilon + M_\epsilon) < -1 \iff a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2.$$

Using arguments similar to those in Gihman and Skorohod ([8] **Part I** Chapter 4 §16) we have the following lemmas. We will only prove the first lemma, keeping in mind that the rest of the lemmas are proved similarly.

**Lemma 3.1.** *If we have the inequalities*

$$a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2 \text{ and } a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2,$$

(hence,  $\lim_{x \rightarrow 1} f(x) = \infty$  and  $\lim_{x \rightarrow 0} f(x) < \infty$ ), then

$$P_{\mathbf{y}_0} \left( \sup_{t>0} \hat{s}_1(t) < 1 \right) = P_{\mathbf{y}_0} \left( \inf_{t>0} \hat{s}_1(t) = 0 \right) = P_{\mathbf{y}_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = 1.$$

*Proof.* We will follow the proof in [8]. Since  $\lim_{y_1 \rightarrow 0} P_{\mathbf{y}_0}(\hat{s}_1(\tau_{y_1 y_2}(y_0)) \geq y_2) = P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t) \geq y_2)$ , we have the equality  $P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t) \geq y_2) = \frac{f(y_0) - f(0)}{f(y_2) - f(0)}$ . Letting  $y_2 \rightarrow 1$ , we conclude that  $P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t) < 1) = 1$ . Also  $P_{\mathbf{y}_0}(\inf_{t>0} \hat{s}_1(t) \leq y_1) \geq P_{\mathbf{y}_0}(\hat{s}_1(\tau_{y_1 y_2}(y_0)) \leq y_1) = \frac{f(y_2) - f(y_0)}{f(y_2) - f(y_1)} \rightarrow 1$  as  $y_2 \rightarrow 1$ .

Showing that  $P_{\mathbf{y}_0}(\limsup_{t \rightarrow \infty} \hat{s}_1(t) \geq y_2) = 0$  will finish the proof. Define  $\tau_q$  to be the first time passage of the process past or to the point  $q < y_0$ . By above we have  $P_{\mathbf{y}_0}(\tau_q < \infty) = 1$ . Furthermore, since  $\tau_q$  is a stopping time  $P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t + \tau_q) \geq y_2) = P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t) \geq y_2 | s(0) = q) = \frac{f(q) - f(0)}{f(y_2) - f(0)}$ . Moreover  $P_{\mathbf{y}_0}(\sup_{t>0} \hat{s}_1(t + \tau_q) \geq y_2) = P_{\mathbf{y}_0}(\sup_{t>\tau_q} \hat{s}_1(t) \geq y_2) \geq P_{\mathbf{y}_0}(\liminf_{t \rightarrow \infty} \hat{s}_1(t) \geq y_2)$ . Thus, taking  $q \rightarrow 0$  we obtain  $P_{\mathbf{y}_0}(\liminf_{t \rightarrow \infty} \hat{s}_1(t) \geq y_2) = 0$ . Therefore  $\liminf_{t \rightarrow \infty} \hat{s}_1(t) = 0$  a.s. and we are done.  $\square$

**Lemma 3.2.** *If we have the inequalities*

$$a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2 \text{ and } a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2,$$

*(hence,  $\lim_{x \rightarrow 1} f(x) = \infty$  and  $\lim_{x \rightarrow 0} f(x) = \infty$ ), then*

$$P_{y_0} \left( \limsup_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = P_{y_0} \left( \liminf_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = 1.$$

**Lemma 3.3.** *If we have the inequalities*

$$a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2 \text{ and } a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2,$$

*(hence,  $\lim_{x \rightarrow 1} f(x) < \infty$  and  $\lim_{x \rightarrow 0} f(x) = \infty$ ), then*

$$P_{y_0} \left( \inf_{t > 0} \hat{s}_1(t) > 0 \right) = P_{y_0} \left( \sup_{t > 0} \hat{s}_1(t) = 1 \right) = P_y \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = 1.$$

**Lemma 3.4.** *If we have the inequalities*

$$a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(\mathbb{R})) + \frac{\epsilon^2}{2}C_2 \text{ and } a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2,$$

*(hence,  $f(x) < \infty \forall x$ ), then*

$$P_{y_0} \left( \sup_{t > 0} \hat{s}_1(t) < 1 \right) = P_{y_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 0 \right) = \frac{f(1) - f(y_0)}{f(1) - f(0)}$$

and

$$P_{y_0} \left( \inf_{t > 0} \hat{s}_1(t) > 0 \right) = P_{y_0} \left( \lim_{t \rightarrow \infty} \hat{s}_1(t) = 1 \right) = \frac{f(y_0) - f(0)}{f(1) - f(0)}.$$

This is very similar to the results found by Fudenberg and Harris [7], with the added or subtracted piece  $\epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2$ . There are cases which the compensated Poisson term does not affect the dynamics of the underlying diffusion model (given by Fudenberg and Harris [7]), in particular, if the differences of the jump sizes are small. However, there are cases which the compensated Poisson term would influence the dynamics of the underlying diffusion. We now give such an example. Recall that  $\nu(\mathbb{R})$  gives us the intensity of the jumps. If  $h_2(x) > 0$  for a significant amount of  $x \in \mathbb{R}$  then  $C_1 < \nu(\mathbb{R})$ . Consider the inequality  $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2}$  where  $\sigma_2 > \sigma_1$  and  $a_{11} - a_{21} < 0$ , and  $h_2(x) > 0$  for a significant amount of  $x \in \mathbb{R}$ . If  $\frac{\sigma_1^2 - \sigma_2^2}{2} - \left( \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2 \right) > 0$ , we have the inequality  $a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \left( \epsilon(C_1 - \nu(\mathbb{R})) - \frac{\epsilon^2}{2}C_2 \right)$ , which changes the dynamics of the stochastic replicator.

**Remark 3.4.** *In most applications to this type of modeling, the Poisson measure and not the compensated Poisson measure is used. This is a simple adjustment of replacing the term  $\epsilon(C_1 - \nu(\mathbb{R}))$  with  $\epsilon C_1$ . It should be noted that this new term has more of an impact to the inequalities of the payoffs and hence the dynamics.*

## 4 Evolutionary Stable Strategies

The rest of the work in this paper will be based off of a paper written by Imhof [11]. Take  $\{e_1, e_2, \dots, e_n\}$  as the standard basis of  $\mathbb{R}^n$ . For the Euclidean norm  $|\cdot|$ , define  $U_\delta(\mathbf{y}') = \{\mathbf{y} \in \Delta_n : |\mathbf{y}' - \mathbf{y}| < \delta\}$  and  $\tau_G = \inf\{t > 0 : \mathbf{s}(t) \in G\}$ , where  $G$  is a Borel set.

For the general  $n$  subpopulation model, we have  $s_i(t)$  is of the form

$$\begin{aligned} ds_i(t) = & s_i(t-) \left[ (As(t-))_i - \sum_j s_j(t-) (As(t-))_j + \sum_j s_j(t-) \sigma_j^2 - s_i(t-) \sigma_i^2 \right. \\ & + \left. \int_{\mathbb{R}} \left( \frac{1 + h_i(x)}{1 + \sum_j s_j(t-) h_j(x)} - 1 + \sum_j s_j(t-) h_j(x) - h_i(x) \right) \nu(dx) \right] dt \\ & + s_i(t-) \left( \sigma_i dW_i(t) - \sum_j s_j(t-) \sigma_j dW_j(t) \right) \\ & + s_i(t-) \int_{\mathbb{R}} \left( \frac{1 + h_i(x)}{1 + \sum_j s_j(t-) h_j(x)} - 1 \right) \tilde{N}(dt, dx). \end{aligned} \quad (9)$$

Define  $\mathbf{h}(x) = (h_1(x), h_2(x), \dots, h_n(x))^T$  and  $\mathbf{W}(t) = (W_1(t), W_2(t), \dots, W_n(t))^T$ . With a little work, one can see that

$$ds(t) = D^1(\mathbf{s}(t-), \mathbf{h}(x))dt + D^2(\mathbf{s}(t-))d\mathbf{W}(t) + \int_{\mathbb{R}} D^3(\mathbf{s}(t-), \mathbf{h}(x))\tilde{N}(dt, dx)$$

where

$$\begin{aligned} D^1(\mathbf{y}, \mathbf{h}(x)) = & [\mathbf{diag}(y_1, \dots, y_n) - \mathbf{y}\mathbf{y}^T] [A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)] \mathbf{y} \\ & + \int_{\mathbb{R}} \left( \mathbf{y}\mathbf{h}(x)^T - \mathbf{diag}(h_1(x), \dots, h_n(x)) \right. \\ & + \left. \frac{1}{1 + \mathbf{y}^T \mathbf{h}(x)} \mathbf{diag}(1 + h_1(x), \dots, 1 + h_n(x)) - \mathbf{diag}(1, \dots, 1) \right) \mathbf{y} \nu(dx), \end{aligned}$$

$$D^2(\mathbf{y}, \mathbf{h}(x)) = [\mathbf{diag}(y_1, \dots, y_n) - \mathbf{y}\mathbf{y}^T] \mathbf{diag}(\sigma_1, \dots, \sigma_n),$$

and

$$D^3(\mathbf{y}, \mathbf{h}(x)) = \left( \frac{1}{1 + \mathbf{y}^T \mathbf{h}(x)} \mathbf{diag}(1 + h_1(x), \dots, 1 + h_n(x)) - \mathbf{diag}(1, \dots, 1) \right) \mathbf{y}.$$

Denote  $\mathcal{A}_J$  as the infinitesimal generator for our process defined by equation (9). By Theorem 2 (Part II Chapter 2 §9) in Gihman and Skorohod [8], we see that

$$\begin{aligned} \mathcal{A}_J f(\mathbf{y}) = & \sum_j \tilde{D}_j^1(\mathbf{y}, \mathbf{h}(x)) \frac{\partial f}{\partial y_j}(\mathbf{y}) + \frac{1}{2} \sum_{j,k} \gamma_{jk}(\mathbf{y}) \frac{\partial^2 f}{\partial y_j \partial y_k}(\mathbf{y}) \\ & + \int_{\mathbb{R}} \left( f(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}) - f(\mathbf{y}) \right) \nu(dx), \end{aligned}$$

where  $D_j^i$  is the  $j^{th}$  coordinate of the function  $D^i$ ,  $\gamma_{jk}(\mathbf{y}) = \sum_l c_{jl}(\mathbf{y}) c_{kl}(\mathbf{y})$  for  $c_{jl}(\mathbf{y}) = \begin{cases} y_j(1 - y_j)\sigma_j, & j = l \\ -y_j y_l \sigma_l & j \neq l \end{cases}$ ,

and

$$\tilde{D}_i^1(\mathbf{y}, \mathbf{h}(x)) := y_i(e_i - \mathbf{y})^T [A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)] \mathbf{y} + \int_{\mathbb{R}} y_i \left( \sum_k y_k h_k(x) - h_i(x) \right) \nu(dx).$$

Before we prove theorems about certain stabilities, we show that our stochastic replicator dynamics evolve close to  $e_k$ , for some  $k \in \{1, \dots, n\}$ . Since the corner points of the simplex are absorbing, this is how we would expect the dynamics to behave.

**Theorem 4.1.** *Take  $\mathbf{s}(t)$  to be an  $n$ -dimensional stochastic replicator dynamic defined by Equation (9), the matrix  $A$  as an arbitrary payoff matrix, and for  $\epsilon > 0$ , define  $\tau_\epsilon := \inf \left\{ t > 0 : s_k(t) \geq 1 - \epsilon \text{ for some } k \in \{1, 2, \dots, n\} \right\}$ . Then for  $\mathbf{y} \in \Delta_n$ ,*

$$\mathbb{E}_{\mathbf{y}}[\tau_\epsilon] < \infty,$$

and

$$P_{\mathbf{y}} \left( \sup_{t \geq 0} \max\{s_1(t), \dots, s_n(t)\} = 1 \right) = 1.$$

*Proof.* We will follow the proof of Theorem 4.3 in Imhof [11]. For  $\alpha > 0$  and  $\mathbf{y} \in \bar{\Delta}_n$  define the positive function  $g(\mathbf{y}) = ne^\alpha - \sum_k e^{\alpha y_k}$ . Define the “new” payoff matrix  $\tilde{A} := A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)$  and the infinitesimal generator  $\mathcal{A}_J$ . Then

$$\begin{aligned} \mathcal{A}_J g(\mathbf{y}) &= -\alpha \sum_k y_k (e_k - \mathbf{y})^T \tilde{A} \mathbf{y} e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \left( \sigma_k^2 (1 - y_k)^2 + \sum_{j \neq k} \sigma_j^2 y_j \right) e^{\alpha y_k} \\ &\quad - \alpha \int_{\mathbb{R}} \sum_k y_k \left( \sum_j y_j h_j(x) - h_k(x) \right) e^{\alpha y_k} \nu(dx) \\ &\quad + \int_{\mathbb{R}} \left[ \sum_k \exp\{\alpha y_k\} - \sum_k \exp \left\{ \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right\} \right] \nu(dx). \end{aligned}$$

For  $\sigma_{\min} := \min\{\sigma_1, \dots, \sigma_n\}$  and a constant  $\beta > 0$  such that  $|(e_k - \mathbf{y})^T \tilde{A} \mathbf{y}| \leq \beta$  for all  $\mathbf{y} \in \Delta_n$  and all  $k \in \{1, \dots, n\}$ , Imhof showed that

$$-\alpha \sum_k y_k (e_k - \mathbf{y})^T \tilde{A} \mathbf{y} e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \leq \alpha \sum_k y_k e^{\alpha y_k} \left[ \beta - \frac{\alpha \sigma_{\min}^2}{2} y_k (1 - y_k)^2 \right]. \quad (10)$$

Furthermore, for  $\kappa_{\max} := \sup_{x \in \mathbb{R}} \max\{h_1(x), \dots, h_n(x)\}$ ,  $\kappa_{\min} := \inf_{x \in \mathbb{R}} \min\{h_1(x), \dots, h_n(x)\}$ , and  $M := \int_{\mathbb{R}} (\kappa_{\max} - \kappa_{\min}) \nu(dx)$ , we have the inequality

$$\alpha \int_{\mathbb{R}} \sum_k y_k \left( -\sum_j y_j h_j(x) + h_k(x) \right) e^{\alpha y_k} \nu(dx) \leq \alpha \sum_k y_k e^{\alpha y_k} M. \quad (11)$$

Recalling the inequality  $-e^x \leq -1 - x$  for  $x > 0$ , we have

$$\begin{aligned} &\int_{\mathbb{R}} \left[ \sum_k \exp\{\alpha y_k\} - \sum_k \exp \left\{ \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right\} \right] \nu(dx) \\ &\leq \sum_k \int_{\mathbb{R}} \left[ \exp\{\alpha y_k\} - 1 - \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \\ &= \sum_k \int_{\mathbb{R}} \left[ \sum_{n=0}^{\infty} \frac{(\alpha y_k)^n}{n!} - 1 - \frac{\alpha y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \end{aligned}$$

$$\begin{aligned}
&= \sum_k \alpha y_k \int_{\mathbb{R}} \left[ \sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} - \frac{1 + h_k(x)}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \\
&\leq \sum_k \alpha y_k \int_{\mathbb{R}} \left[ \sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} \right] \nu(dx) \\
&= \sum_k \alpha y_k \exp\{\alpha y_k\} \int_{\mathbb{R}} \left[ \exp\{-\alpha y_k\} \sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} \right] \nu(dx) \\
&\leq \alpha \sum_k y_k \exp\{\alpha y_k\} \nu(\mathbb{R}).
\end{aligned} \tag{12}$$

Collecting Equations (10), (11), and (12), we see that

$$\mathcal{A}_J g(\mathbf{y}) \leq \alpha \sum_k y_k e^{\alpha y_k} \left[ \left( \beta + M + \nu(\mathbb{R}) \right) - \frac{\alpha \sigma_{\min}^2}{2} y_k (1 - y_k)^2 \right]$$

Now for an arbitrarily small  $\epsilon > 0$ , choose  $\alpha > 0$  large enough that  $\alpha \frac{\sigma_{\min}^2}{2} y(1 - y)^2 \geq (\beta + M + \nu(\mathbb{R}))n + 1$  for all  $y \in [\frac{1}{n}, 1 - \epsilon]$ . Furthermore, take  $\mathbf{y} \in \Delta_n$  such that  $y_i \leq 1 - \epsilon$  for all  $i$ . For our  $\mathbf{y}$ , there is at least one  $y_k$  such that  $y_k \geq \frac{1}{n}$  and hence

$$\begin{aligned}
\mathcal{A}_J g(\mathbf{y}) &\leq \alpha \left( \beta + M + \nu(\mathbb{R}) \right) \sum_{k: y_k < 1/n} y_k e^{\alpha y_k} + \alpha \sum_{k: y_k \geq 1/n} y_k e^{\alpha y_k} \left( - (n - 1) \left( \beta + M + \nu(\mathbb{R}) \right) - 1 \right) \\
&\leq \alpha \left( \beta + M + \nu(\mathbb{R}) \right) (n - 1) \frac{e^{\alpha/n}}{n} + \alpha \frac{e^{\alpha/n}}{n} \left( - (n - 1) \left( \beta + M + \nu(\mathbb{R}) \right) - 1 \right) \\
&= -\alpha \frac{e^{\alpha/n}}{n}.
\end{aligned}$$

Now by Dynkin's formula for every finite  $T$ ,

$$\begin{aligned}
0 \leq \mathbb{E}_{\mathbf{y}} \left[ g(\mathbf{s}(\tau_{\epsilon} \wedge T)) \right] &= g(\mathbf{y}) + \mathbb{E}_{\mathbf{y}} \left[ \int_0^{\tau_{\epsilon} \wedge T} \mathcal{A}_J g(\mathbf{s}(t)) dt \right] \\
&\leq n e^{\alpha} - \alpha \frac{e^{\alpha/n}}{n} \mathbb{E}_{\mathbf{y}} [\tau_{\epsilon} \wedge T].
\end{aligned}$$

Therefore, by the monotone convergence theorem, letting  $T \rightarrow \infty$  yields the inequality  $\mathbb{E}_{\mathbf{y}} [\tau_{\epsilon}] \leq n^2 \frac{e^{\alpha}}{\alpha}$ .

Finally, take  $\epsilon = 1/m$  for  $m \in \mathbb{N}$ . Then  $P_{\mathbf{y}} \left( \sup_{t > 0} \max\{s_1(t), \dots, s_n(t)\} \geq 1 - 1/m \right) = 1$ , and therefore

$$1 = P_{\mathbf{y}} \left( \bigcap_{m=1}^{\infty} \left\{ \sup_{t > 0} \max\{s_1(t), \dots, s_n(t)\} \geq 1 - 1/m \right\} \right) = P_{\mathbf{y}} \left( \sup_{t > 0} \max\{s_1(t), \dots, s_n(t)\} = 1 \right).$$

□

Take  $A$  to be a payoff matrix for a game. A strategy  $\mathbf{p} \in \overline{\Delta}_n$  is called an evolutionary stable strategy if:  $\mathbf{q}^T A \mathbf{p} \leq \mathbf{p}^T A \mathbf{p}$  for all  $\mathbf{q} \in \overline{\Delta}_n$ ; and for  $\mathbf{q} \in \overline{\Delta}_n$  where  $\mathbf{q} \neq \mathbf{p}$  and  $\mathbf{q}^T A \mathbf{p} = \mathbf{p}^T A \mathbf{p}$ , we have that  $\mathbf{q} \cdot A \mathbf{q} < \mathbf{p} \cdot A \mathbf{q}$ . Imhof [11] noted that the concept of an evolutionary stable strategy, although a stronger notion than a Nash equilibria, is not strong enough to hold by itself in a stochastic setting. The author adjusted for this weakness by assuming the

payoff matrix is conditional negative definite, which is defined below. From these assumptions, the author was then able to show conditions for stability near an evolutionary stable strategy. We utilize this technique developed by the author.

**Definition 4.1.** A matrix  $A$  is said to be conditionally negative definite if for  $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  where  $\mathbf{1}^T \mathbf{y} = 0$ , we have

$$\mathbf{y}^T A \mathbf{y} < 0.$$

**Lemma (Imhof [11]).** Suppose that  $A$  is an  $n \times n$  ( $n \geq 2$ ) conditionally negative definite matrix, define  $\bar{A} = \frac{1}{2}(A + A^T)$  and let  $\lambda_2$  be the second largest eigenvalue of

$$D := \bar{A} - \frac{1}{n} \bar{A} \mathbf{1} \mathbf{1}^T - \frac{1}{n} \mathbf{1} \mathbf{1}^T \bar{A} + \frac{\mathbf{1}^T \bar{A} \mathbf{1}}{n} \mathbf{1} \mathbf{1}^T.$$

Then

$$\max_{\substack{\mathbf{x}^T \mathbf{1} = 0 \\ \mathbf{x} \neq \mathbf{0}}} \frac{\mathbf{x}^T D \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \max_{\substack{\mathbf{x}^T \mathbf{1} = 0 \\ \mathbf{x} \neq \mathbf{0}}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \lambda_2 < 0.$$

Define  $d(\mathbf{y}, \mathbf{p}) = \sum_j p_j \log(p_j/y_j)$ , where  $\log(p_j/y_j) = 0$  if  $p_j = 0$  or  $y_j = 0$ . This is the known as the Kullback-Leibler distance. We are now ready to state and prove the theorem.

**Theorem 4.2.** Take  $\mathbf{s}(t)$  as above,  $\mathbf{p} \in \Delta_n$  an ESS for our payoff matrix  $A$ , which we assume is conditionally negative definite, and  $\lambda_2$  as the second largest eigenvalue of  $D$ . Define

$$\begin{aligned} \kappa_J^2 &= \frac{1}{2} \sum_j p_j \sigma_j^2 - \frac{1}{2 \sum_j \sigma_j^{-2}} + \int_{\mathbb{R}} \max_k h_k(x) \nu(dx) \\ &+ \sum_j p_j \int_{\mathbb{R}} \log \left( \frac{1 + \max_k h_k(x)}{1 + h_j(x)} \right) \nu(dx) - \sum_j p_j \int_{\mathbb{R}} h_j(x) \nu(dx) \end{aligned}$$

and assume that  $0 < \kappa_J < \frac{n}{n-1} \sqrt{|\lambda_2|} \min_{1 \leq j \leq n} p_j$ . Furthermore, assume that  $\int_{\mathbb{R}} \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) < 0$  for all  $\mathbf{y} \in \Delta_n$ . Then for  $\delta > 0$  such that  $\delta^2 > \kappa_J^2/|\lambda_2|$ ,  $\mathbf{y} \in \Delta_n$ , and  $t > 0$ , we have the inequalities

$$\mathbb{E}_{\mathbf{y}}[\tau_{\bar{U}_\delta(\mathbf{p})}] \leq \frac{d(\mathbf{y}, \mathbf{p})}{|\lambda_2| \delta^2 - \kappa_J^2}. \quad (13)$$

and

$$\mathbb{E}_{\mathbf{y}} \left[ \frac{1}{t} \int_0^t |\mathbf{s}(u) - \mathbf{p}|^2 du \right] \leq \frac{1}{|\lambda_2|} \left( \frac{d(\mathbf{y}, \mathbf{p})}{t} + \kappa_J^2 \right). \quad (14)$$

Lastly, an invariant measure  $\pi(\cdot)$  of the stochastic replicator dynamic, exists, is unique, and

$$\pi(U_\delta(\mathbf{p})) \geq 1 - \frac{\kappa_J^2}{|\lambda_2| \delta^2}. \quad (15)$$

*Proof.* Our proof will very closely follow the one given by Imhof [11]. For  $\mathbf{p} \in \Delta_n$  an ESS for  $A$ , define the function  $v(\mathbf{y}) = \sum_j p_j \log(p_j/y_j)$ . Applying  $\mathcal{A}_J$  (the infinitesimal generator) to  $v$ , we see that

$$\begin{aligned} \mathcal{A}_J v(\mathbf{y}) &= - \sum_j p_j (e_i - \mathbf{y})^T \left[ A - \text{diag}(\sigma_1^2, \dots, \sigma_n^2) \right] \mathbf{y} + \frac{1}{2} \sum_j p_j \left( \sigma_j^2 - 2y_j \sigma_j^2 + \sum_k y_k^2 \sigma_k^2 \right) \\ &\quad - \sum_j p_j \int_{\mathbb{R}} \left( h_j(x) - \sum_k y_k h_k(x) \right) \nu(dx) + \sum_j p_j \int_{\mathbb{R}} \log \left( \frac{1 + \sum_k y_k h_k(x)}{1 + h_j(x)} \right) \nu(dx) \\ &= (\mathbf{y} - \mathbf{p})^T A \mathbf{y} - \frac{1}{2} \sum_j y_j^2 \sigma_j^2 + \frac{1}{2} \sum_j p_j \sigma_j^2 - \sum_j p_j \int_{\mathbb{R}} \left( h_j(x) - \sum_k y_k h_k(x) \right) \nu(dx) \\ &\quad + \sum_j p_j \int_{\mathbb{R}} \log \left( \frac{1 + \sum_k y_k h_k(x)}{1 + h_j(x)} \right) \nu(dx) \\ &\leq (\mathbf{y} - \mathbf{p})^T A \mathbf{y} - \frac{1}{2} \sum_j y_j^2 \sigma_j^2 + \frac{1}{2} \sum_j p_j \sigma_j^2 + \int_{\mathbb{R}} \max_k h_k(x) \nu(dx) - \sum_j p_j \int_{\mathbb{R}} h_j(x) \nu(dx) \\ &\quad + \sum_j p_j \int_{\mathbb{R}} \log \left( \frac{1 + \max_k h_k(x)}{1 + h_j(x)} \right) \nu(dx). \end{aligned}$$

In the proof of Theorem 2.1 in [11], Imhof showed that  $(\mathbf{y} - \mathbf{p})^T A \mathbf{y} \leq \lambda_2 |\mathbf{y} - \mathbf{p}|^2$  and  $-\frac{1}{2} \sum_j y_j^2 \sigma_j^2 \leq -\frac{1}{2 \sum_j \sigma_j^{-2}}$ .

Thus, for  $y \in \Delta_n$ ,

$$\mathcal{A}_J v(\mathbf{y}) \leq \lambda_2 |\mathbf{y} - \mathbf{p}|^2 + \kappa_J^2.$$

Our assumption  $\delta^2 > \kappa_J^2/|\lambda_2|$  tells us for  $\mathbf{y} \in \Delta_n \setminus U_\delta(\mathbf{p})$ ,  $\mathcal{A}_J v(\mathbf{y}) \leq \lambda_2 \delta^2 + \kappa_J^2$ . By Itô's lemma, the process  $v(\mathbf{s}(t)) - (\lambda_2 \delta^2 + \kappa_J^2)t$  is a local supermartingale on the interval  $[0, \tau_{\overline{U}_\delta(\mathbf{p})})$ . Therefore  $v(\mathbf{y}) \geq (|\lambda_2| \delta^2 - \kappa_J^2) \mathbb{E}_{\mathbf{y}}[\tau_{\overline{U}_\delta(\mathbf{p})}]$ , which shows Equation (13). The strong Markov property tells us that the stochastic replicator dynamic is recurrent in the set  $U_\delta(\mathbf{p})$ . Furthermore, by choosing a  $\delta_0 > 0$  where  $\kappa_J < \delta_0 < \frac{n}{n-1} \sqrt{|\lambda_2|} \min_{1 \leq j \leq n} p_j$ , Imhoff [11] in Theorem 2.1 showed that  $\overline{\Delta}_n \setminus \{\Delta_n \cap \overline{U}_\delta(\mathbf{p})\} = \emptyset$ . Thus, our process never hits the boundary and we are able to pick any  $\delta > 0$  for which the inequality holds.

Now define  $\tau_k = \inf\{t > 0 : v(\mathbf{s}(t)) = k\}$ , where  $k > v(\mathbf{y})$ . Applying Dynkin's formula we see that

$$\begin{aligned} 0 &\leq \mathbb{E}_{\mathbf{y}} \left[ v(\mathbf{s}(t \wedge \tau_k)) \right] = v(\mathbf{y}) + \mathbb{E}_{\mathbf{y}} \left[ \int_0^{t \wedge \tau_k} \mathcal{A}_J v(\mathbf{s}(u)) du \right] \\ &\leq v(\mathbf{y}) + \lambda_2 \mathbb{E}_{\mathbf{y}} \left[ \int_0^{t \wedge \tau_k} |\mathbf{s}(u) - \mathbf{p}|^2 du \right] + \kappa_J^2 \mathbb{E}_{\mathbf{y}}[t \wedge \tau_k] \end{aligned}$$

Since  $t \wedge \tau_k \rightarrow t$  as  $k \rightarrow \infty$ , the bounded convergence theorem yields Equation (14).

Finally, to show Equation (15) we need to show that the transition probabilities converge in total variation to an invariant measure (which makes this measure unique). To accomplish this task we will apply Theorem 5.2 in Down et al [4]. In order to satisfy the hypotheses of the theorem, we need to show that our process is  $\psi$ -irreducible (page 1674 [4]) and aperiodic (page 1675 [4]). To show the  $\psi$ -irreducible condition, we define the Borel measure  $\psi(O) = M(O \cap U_\delta(\mathbf{p}))$ , where  $M$  is the Lebesgue measure, and  $\eta_O := \int_0^\infty \mathbf{1}_{\{\mathbf{s}(t) \in O\}} dt$  (the occupancy time). Since we know our process is recurrent in  $U_\delta(\mathbf{p})$ , if  $\psi(O) > 0$  then  $\mathbb{E}_{\mathbf{y}}[\eta_O] > 0$ .

To show the aperiodic condition we need to find a small Borel set  $B$  and a time  $T$  such that  $P_{\mathbf{y}}(t, B) > 0$  for all  $t \geq T$  and all  $\mathbf{y} \in B$ . A clear candidate for  $B$  is the set  $U_\delta(\mathbf{p})$ . Before we show this conditions holds, we will note that since the Poisson measure is generated by a Lévy process, (and so the initial condition for Lévy process is Dirac measure  $\delta_0$ ), independent of all the Wiener processes, the jumps are only dependent on time.

To show that this condition holds, we will follow the proof of Claim 1 given in [15]. Since  $\nu(\mathbb{R}) < \infty$ , we may rewrite  $s(t)$  as

$$\mathbf{s}(t) = \mathbf{y} + \int_0^t \hat{D}^1(\mathbf{s}(t-), \mathbf{h}(x)) dt + \int_0^t D^2(\mathbf{s}(t-)) d\mathbf{W}(t) + \int_0^t \int_{\mathbb{R}} D^3(\mathbf{s}(t-), \mathbf{h}(x)) N(dt, dx),$$

where

$$\hat{D}^1(\mathbf{y}, \mathbf{h}(x)) = [\mathbf{diag}(y_1, \dots, y_n) - \mathbf{y}\mathbf{y}^T] [A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)] \mathbf{y} + \int_{\mathbb{R}} (\mathbf{y}\mathbf{h}(x)^T - \mathbf{diag}(h_1(x), \dots, h_n(x))) \nu(dx).$$

For the finite interval  $[0, t']$  (for any  $t' > 0$ ), there is a positive  $P_{\mathbf{y}}$  probability that a jump does not occur. On this event,  $s(t)$  agrees with the process

$$\mathbf{l}(t) = \mathbf{y} + \int_0^t \hat{D}^1(\mathbf{l}(t), \mathbf{h}(x)) dt + \int_0^t D^2(\mathbf{l}(t)) d\mathbf{W}(t).$$

Thus, considering Theorem 2.1 in Imhoff [11], the condition holds.

Lastly, we need to show that  $\mathcal{A}_J V(\cdot) \leq -cV(\cdot) + b\mathbf{1}_{U_\delta(\mathbf{p})}(\cdot)$  where  $V \geq 1$ ,  $V \in D(\mathcal{A}_J)$ , and  $c, b > 0$ . Define  $V(\mathbf{y}) = K + \prod_l y_l^{-p_l}$ , where  $K$  is a positive constant which will later be determined. So

$$\begin{aligned} \mathcal{A}_J V(\mathbf{y}) &= -\sum_i p_i \left[ (e_i - \mathbf{y})^T [A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)] \mathbf{y} + \int_{\mathbb{R}} \left( \sum_j y_j h_j(x) - h_i(x) \right) \nu(dx) \right] \prod_l y_l^{-p_l} \\ &\quad + \frac{1}{2} \sum_i p_i(p_i + 1) \left[ (1 - 2y_i)\sigma_i^2 + \sum_j y_j^2 \sigma_j^2 \right] \prod_l y_l^{-p_l} + \frac{1}{2} \sum_i \sum_{i \neq k} p_i p_k \left[ \sum_j y_j^2 \sigma_j^2 - y_i \sigma_i^2 - y_k \sigma_k^2 \right] \prod_l y_l^{-p_l} \\ &\quad + \int_{\mathbb{R}} \left( V(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}) - V(\mathbf{y}) \right) \nu(dx) \\ &= (\mathbf{y} - \mathbf{p})^T A \mathbf{y} \cdot \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \sum_j (p_j - y_j) h_j(x) \nu(dx) \cdot \prod_l y_l^{-p_l} + \sum_j y_j (p_j - y_j) \sigma_j^2 \cdot \prod_l y_l^{-p_l} \\ &\quad + \sum_i p_i \left[ (1 - 2y_i)\sigma_i^2 + \sum_j y_j^2 \sigma_j^2 \right] \prod_l y_l^{-p_l} - \frac{1}{2} \sum_i p_i \sum_{k \neq i} p_k \left[ (1 - 2y_i)\sigma_i^2 + \sum_j y_j^2 \sigma_j^2 \right] \prod_l y_l^{-p_l} \\ &\quad + \frac{1}{2} \sum_i \sum_{i \neq k} p_i p_k \left[ \sum_j y_j^2 \sigma_j^2 - y_i \sigma_i^2 - y_k \sigma_k^2 \right] \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \left( V(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}) - V(\mathbf{y}) \right) \nu(dx) \\ &= (\mathbf{y} - \mathbf{p})^T A \mathbf{y} \cdot \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \sum_j (p_j - y_j) h_j(x) \nu(dx) \cdot \prod_l y_l^{-p_l} + \sum_j p_i (1 - y_j) \sigma_j^2 \cdot \prod_l y_l^{-p_l} \\ &\quad - \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left[ (1 - y_i)\sigma_i^2 + y_k \sigma_k^2 \right] \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \left( V(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}) - V(\mathbf{y}) \right) \nu(dx) \\ &\leq \left( \lambda_2 |\mathbf{p} - \mathbf{y}|^2 + \sum_j p_i (1 - y_j) \sigma_j^2 + \int_{\mathbb{R}} \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) \right. \\ &\quad \left. - \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left[ (1 - y_i)\sigma_i^2 + y_k \sigma_k^2 \right] \right) \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \frac{1 + \max_j h_j(x)}{1 + \min_j h_j(x)} \nu(dx) := C(\mathbf{y}) \prod_l y_l^{-p_l} + \varsigma, \end{aligned}$$

for

$$C(\mathbf{y}) = \lambda_2 |\mathbf{p} - \mathbf{y}|^2 + \sum_j p_i (1 - y_j) \sigma_j^2 + \int_{\mathbb{R}} \left( \sum_j (p_j - y_j) h_j(x) - 1 \right) \nu(dx) - \frac{1}{2} \sum_i \sum_{k \neq i} p_i p_k \left[ (1 - y_i)\sigma_i^2 + y_k \sigma_k^2 \right]$$

and

$$\varsigma = \int_{\mathbb{R}} \frac{1 + \max_j h_j(x)}{1 + \min_j h_j(x)} \nu(dx).$$

To finish the inequality, we note that



$$\begin{aligned}
C(\mathbf{y}) \prod_l y_l^{-p_l} + \varsigma &= \left( \frac{C(\mathbf{y}) \prod_l y_l^{-p_l}}{V(\mathbf{y})} + \frac{\varsigma}{V(\mathbf{y})} \right) V(\mathbf{y}) \\
&= \left( \frac{C(\mathbf{y}) \prod_l y_l^{-p_l}}{K + \prod_l y_l^{-p_l}} + \frac{\varsigma}{K + \prod_l y_l^{-p_l}} \right) V(\mathbf{y}) \leq \left( C(\mathbf{y}) + \frac{\varsigma}{K} \right) V(\mathbf{y}).
\end{aligned}$$

By our assumptions  $C(\mathbf{y}) < 0$  for  $\mathbf{y} \in \Delta_n \setminus U_\delta(\mathbf{p})$ . Thus, taking  $K$  large enough so that  $C(\mathbf{y}) + \frac{\varsigma}{K} < 0$  for all  $\mathbf{y} \in \Delta_n \setminus U_\delta(\mathbf{p})$  and  $V \geq 1$ , we are able to find a constants  $c, b > 0$  such that  $\mathcal{A}_J V(\mathbf{y}) \leq -cV(\mathbf{y}) + b\mathbf{1}_{U_\delta(\mathbf{p})}(\mathbf{y})$  holds for all  $\mathbf{y} \in \Delta_n$ .

Defining  $O^C := \Delta_n \setminus O$  and  $\pi(\cdot)$  as the invariant measure, we have

$$\begin{aligned}
\pi\left(\overline{U}_\delta(\mathbf{p})^C\right) &= \lim_{t \rightarrow \infty} \mathbb{E}_{\mathbf{y}} \left[ \frac{1}{t} \int_0^t \mathbf{1}_{\overline{U}_\delta(\mathbf{p})^C}(\mathbf{s}(u)) du \right] \\
&\leq \lim_{t \rightarrow \infty} \mathbb{E}_{\mathbf{y}} \left[ \frac{1}{t} \int_0^t \frac{|\mathbf{s}(u) - \mathbf{p}|^2}{\delta^2} du \right] \leq \frac{\kappa_J^2}{|\lambda_2| \delta^2},
\end{aligned}$$

and therefore Equation (15) follows.  $\square$

## 5 Strict Nash and Stochastic Stability in the Presence of Continuous and Random Jumps

We call a strategy  $\mathbf{p} \in \overline{\Delta}_n$  a strict Nash Equilibria if for  $\mathbf{q} \in \overline{\Delta}_n$  such that  $\mathbf{q} \neq \mathbf{p}$ ,  $\mathbf{q}^T A \mathbf{p} < \mathbf{p}^T A \mathbf{p}$ . In this section we examine strict Nash Equilibria and how random jumps and white noise affect the stability of replicator dynamics. Throughout this section take pure strategy  $S_k$  as a strict Nash Equilibria, i.e.,  $a_{kk} > a_{jk}$  for all  $j \neq k$ . Since the characteristics of the jumps are able to impact stability we define the functions  $\psi_{min}^k(x) := \min_{j \neq k} h_j(x)$ ,  $\psi_{max}^k(x) := \max_{j \neq k} h_j(x)$ , and  $\psi_{max}(x) := \max_j h_j(x)$ . Furthermore, from these functions we define the integrals  $I_1^k := \int_{\mathbb{R}} \frac{(h_k(x) - \psi_{min}^k(x))^2}{1 + \psi_{max}(x)} \nu(dx)$  and  $I_2^k := \int_{\mathbb{R}} \frac{\psi_{min}^k(x)^2 + h_k - (1 + h_k(x)) \psi_{max}^k(x)}{1 + \psi_{max}(x)} \nu(dx)$ . Finally, for the purposes of the theorem below, for  $\tilde{A}$  defined in Theorem 4.1, define  $\beta := \max\{|\tilde{a}_{ji}| : 1 \leq j, i \leq n\}$ .

**Theorem 5.1.** *Take the matrix  $A$  and the process  $\mathbf{s}(t)$  defined in Equation (9). Assume that for the pure strategy  $S_k$  and the corresponding variance  $\sigma_k^2$ , we have the inequality  $a_{kk} > a_{jk} + \sigma_j^2$  for all  $j \neq k$  (hence  $S_k$  is a strict Nash equilibrium),  $h_i(x)$  is nonnegative for all  $i$ , and  $-2\beta + I_2^k \geq 0$ . Furthermore, for  $\alpha > 0$ , where  $\alpha + a_{jk} < a_{kk} - \sigma_k^2$ , assume  $\alpha + 2\beta - I_1^k \geq 0$ . Then for every  $\delta > 0$ ,*

$$P_{\tilde{\mathbf{y}}} \left( \lim_{t \rightarrow \infty} \mathbf{s}(t) = e_k \right) \geq 1 - \frac{1 - \tilde{y}_k}{1 - \delta}. \quad (16)$$

*Proof.* Take  $\tilde{A}$  as in Theorem 4.1. Applying the infinitesimal generator  $\mathcal{A}_J$  to our Lyapunov function  $g(\mathbf{y}) = 1 - y_k$ , we have

$$\begin{aligned}
\mathcal{A}_J g(\mathbf{y}) &= -y_k (e_k - \mathbf{y})^T \tilde{A} \mathbf{y} \\
&+ -y_k \int_{\mathbb{R}} \left( \frac{1 + h_k(x)}{1 + \sum_j y_j h_j(x)} + \sum_j y_j h_j(x) - h_k(x) - 1 \right) \nu(dx) \\
&+ \int_{\mathbb{R}} \left( 1 - \left( \frac{y_k (1 + h_k(x))}{1 + \sum_j y_j h_j(x)} - y_k + y_k \right) + 1 - y_k \right) \nu(dx) \\
&= -y_k (e_k - \mathbf{y})^T \tilde{A} \mathbf{y} \\
&+ -y_k \int_{\mathbb{R}} \left( \frac{h_k(x) - \sum_j y_j h_j(x) + \left( \sum_j y_j h_j(x) \right)^2 - h_k(x) \sum_j y_j h_j(x)}{1 + \sum_j y_j h_j(x)} \right) \nu(dx)
\end{aligned}$$

Imhof [11] showed that

$$-y_k(e_k - \mathbf{y})^T \tilde{\mathbf{A}}\mathbf{y} \leq -y_k[(\alpha + 2\beta)y_k - 2\beta]g(\mathbf{y}), \quad (17)$$

so we must focus on the integral term.

For  $\mathbf{y} \in \Delta_n$ , we have  $1 + \sum_j y_j h_j(x) \geq 1 + \min_j h_j(x) > 0$  by Assumption 2.1. Hence, we may just focus on the numerator of the integrand to find an inequality. Using the  $\psi^k$  functions defined in the beginning of the section, we determine that

$$\begin{aligned} & \int_{\mathbb{R}} \left[ h_k(x) - \sum_j y_j h_j(x) + \left( \sum_j y_j h_j(x) \right)^2 - h_k(x) \sum_j y_j h_j(x) \right] \nu(dx) \\ &= \int_{\mathbb{R}} \left[ h_k(x) - y_k h_k(x) - \sum_{j \neq k} y_j h_j(x) + \left( y_k h_k(x) + \sum_{j \neq k} y_j h_j(x) \right)^2 - y_k h_k(x)^2 - h_k(x) \sum_{j \neq k} y_j h_j(x) \right] \nu(dx) \\ &\geq \int_{\mathbb{R}} \left[ h_k(x) - y_k h_k(x) - \psi_{max}^k(x) \sum_{j \neq k} y_j + \left( y_k h_k(x) + \psi_{min}^k(x) \sum_{j \neq k} y_j \right)^2 - y_k h_k(x)^2 - h_k(x) \psi_{max}^k(x) \sum_{j \neq k} y_j \right] \nu(dx) \\ &= \int_{\mathbb{R}} \left[ h_k(x) - y_k h_k(x) - \psi_{max}^k(x)(1 - y_k) + \left( y_k h_k(x) + \psi_{min}^k(x)(1 - y_k) \right)^2 - y_k h_k(x)^2 - h_k(x) \psi_{max}^k(x)(1 - y_k) \right] \nu(dx) \\ &= \int_{\mathbb{R}} \left[ h_k(x) - y_k h_k(x) - \psi_{max}^k(x)(1 - y_k) + y_k^2 h_k(x)^2 + 2y_k h_k(x) \psi_{min}^k(x)(1 - y_k) + \psi_{min}^k(x)^2 (1 - y_k)^2 \right. \\ &\quad \left. - y_k h_k(x)^2 - h_k(x) \psi_{max}^k(x)(1 - y_k) \right] \nu(dx) \\ &= \int_{\mathbb{R}} \left[ h_k - \psi_{max}^k(x) + -y_k h_k(x)^2 + 2y_k h_k(x) \psi_{min}^k(x) + \psi_{min}^k(x)^2 (1 - y_k) - h_k(x) \psi_{max}^k(x) \right] \nu(dx) \cdot (1 - y_k) \\ &= \int_{\mathbb{R}} \left[ -y_k \left( h_k(x)^2 - 2h_k(x) \psi_{min}^k(x) + \psi_{min}^k(x)^2 \right) + \psi_{min}^k(x)^2 + h_k(x) - (1 + h_k(x)) \psi_{max}^k(x) \right] \nu(dx) \cdot (1 - y_k) \\ &= \int_{\mathbb{R}} \left[ -y_k \left( h_k(x) - \psi_{min}^k(x) \right)^2 + \psi_{min}^k(x)^2 + h_k(x) - (1 + h_k(x)) \psi_{max}^k(x) \right] \nu(dx) \cdot (1 - y_k). \end{aligned}$$

Hence we have the inequality

$$-y_k \int_{\mathbb{R}} \left( \frac{\left( \sum_j y_j h_j(x) \right)^2 - h_k(x) \sum_j y_j h_j(x)}{1 + \sum_j y_j h_j(x)} \right) \nu(dx) \leq -y_k \left[ -I_1^k y_k + I_2^k \right] g(\mathbf{y}). \quad (18)$$

Thus by Equations (17) and (18)

$$\mathcal{A}_J g(\mathbf{y}) \leq -y_k \left[ (\alpha + 2\beta - I_1^k) y_k - (2\beta - I_2^k) \right] g(\mathbf{y}) := \hat{g}(\mathbf{y}).$$

With our assumptions we have  $\hat{g}(\mathbf{y}) \leq 0$  for every  $\mathbf{y} \in \Delta_n$ . For an arbitrary  $\delta > 0$ , define  $V_\delta = \{\mathbf{y} \in \Delta_n : y_k > \delta\}$ , and  $\tau_{V_\delta}$  as the first time the process leaves  $V_\delta$ . Then  $g(\mathbf{s}(t \wedge \tau_{V_\delta}))$  is a local supermartingale, and thus for  $\tilde{\mathbf{y}} \in V_\delta$ ,

$$P_{\tilde{\mathbf{y}}} \left( \sup_{0 \leq t < \infty} g(\mathbf{s}(t \wedge \tau_{V_\delta})) \geq 1 - \delta \right) \leq \frac{g(\tilde{\mathbf{y}})}{1 - \delta}$$

which implies

$$P_{\tilde{\mathbf{y}}} \left( \sup_{0 \leq t < \infty} g(\mathbf{s}(t \wedge \tau_{V_\delta})) < 1 - \delta \right) \geq 1 - \frac{g(\tilde{\mathbf{y}})}{1 - \delta}.$$

Notice that for  $\epsilon > 0$ , there is a  $d > 0$  such that  $\hat{g}(\mathbf{y}) \leq -d$  on  $V_\delta \setminus U_\epsilon(e_k)$ . Therefore, applying the logic given in the proof of Theorem 2 in Kushner [14], we are able to conclude the theorem.  $\square$

**Remark 5.1.** If for all  $x$  we have  $|\psi_{\max}(x) - \psi_{\min}(x)|$  is small and  $h_k(x)$  is sufficiently larger than  $\psi_{\max}(x)$ , then  $I_2^k \geq 0$ . Hence, if  $2\beta$  is small enough, we have  $-2\beta + I_2^k \geq 0$ . Furthermore, if  $I_1^k < I_2^k$ , it is very likely that  $2\beta - I_1^k \geq 0$ , and so  $\alpha + 2\beta - I_1^k \geq 0$ . However, if  $2\beta - I_1^k < 0$ , then  $\alpha$  could be large enough so that  $\alpha + 2\beta - I_1^k \geq 0$ . A case like this is possible with  $\sigma_k^2$  small and  $\min\{|a_{kk} - a_{jk}| : 1 \leq j \leq n \text{ and } j \neq k\}$  relatively large.

**Corollary 5.1.** Assume that for the pure strategy  $S_j$  and the corresponding variance  $\sigma_j^2$ , we have the inequality  $a_{kk} > a_{jk} + \sigma_j^2$  for all  $j \neq k$ ,  $h_i(x)$  is nonnegative for all  $i$ ,  $-2\beta + I_2^k < 0$ ,  $\alpha + 2\beta - I_1^k > 0$ , and  $I_1^k \leq I_2^k + \alpha/2$ . Then there exists a neighborhood of  $e_k$ , say  $V \subset \Delta_n$ , such that for any  $\tilde{\mathbf{y}} \in V$  and  $\lambda = \sup_{\mathbf{y} \in \partial V} |1 - y_k|$ ,

$$P_{\tilde{\mathbf{y}}} \left( \lim_{t \rightarrow \infty} \mathbf{s}(t) = e_k \right) \geq 1 - \frac{1 - \tilde{y}_k}{1 - \lambda}.$$

*Proof.* From Theorem 5.1, we have the inequality  $\mathcal{A}_j g(\mathbf{y}) \leq -y_k \left[ (\alpha + 2\beta - I_1^k) y_k - (2\beta - I_2^k) \right] g(\mathbf{y})$ . Define  $V = \left\{ \mathbf{y} \in \Delta_n : y_k > \frac{1}{2} \frac{\alpha + 4\beta - 2I_2^k}{\alpha + 2\beta - I_1^k} \right\}$ . Since  $-y_k \left[ (\alpha + 2\beta - I_1^k) y_k - (2\beta - I_2^k) \right] g(\mathbf{y})$  is negative for  $\mathbf{y} \in V$ , the rest of the proof follows the proof given in Theorem 5.1.  $\square$

## 6 Dominated Strategies

In the past sections we have shown conditions where our process will converge to a pure strategy but we have not considered the possibility of a pure strategy which there is no possibility of convergence. We say that a strategy  $\mathbf{q}$  is dominated by  $\mathbf{p}$  if for any strategy you play against your better payoff comes from strategy  $\mathbf{p}$ , i.e.,  $\mathbf{q}^T \mathbf{A} \mathbf{p}' \leq \mathbf{p}^T \mathbf{A} \mathbf{p}'$  for all  $\mathbf{p}' \in \bar{\Delta}_n$ . In the theorem below, we show under appropriate stochastic perturbations a dominated pure strategy becomes extinct.

**Theorem 6.1.** Let the pure strategy  $S_k$  be dominated by the mixed strategy  $\mathbf{p} \in \bar{\Delta}_n$ . For our payoff matrix  $A$  define  $K_1 = \min_{\mathbf{q} \in \bar{\Delta}_n} \{\mathbf{p}^T A \mathbf{q} - e_k^T A \mathbf{q}\}$ ,  $K_2 = -\frac{\sigma_k^2}{2} + \frac{1}{2} \sum_j p_j \sigma_j^2$ , and define  $\sigma_{\max} = \max\{\sigma_1, \dots, \sigma_n\}$ . Suppose that  $K_2 < K_1$ , and  $h_k(x) \leq \sum_j p_j h_j(x)$  for all  $x \in \mathbb{R}$ . Then for every  $\mathbf{y} \in \bar{\Delta}_n$ ,

$$P_{\mathbf{y}} \left( s_k(t) = o \left( \exp \left\{ t \int_{\mathbb{R}} \left( h_k(x) - \sum_j p_j h_j(x) \right) \nu(dx) - (K_1 - K_2)t + 3\sigma_{\max} \sqrt{t \log \log t} \right\} \right) \right) = 1.$$

*Proof.* We will proceed as in Theorem 3.1 (Imhof [11]). Working from the dominating mixed strategy  $\mathbf{p}$ , define  $G(t) = \log(s_k(t)) - \sum_j p_j \log(s_j(t))$ . Itô's lemma yields

$$\begin{aligned} G(t) = & G(0) + \int_0^t \left( e_k^T A \mathbf{s}(u) - \mathbf{p}^T A \mathbf{s}(u) - \frac{\sigma_k^2}{2} + \frac{1}{2} \sum_j p_j \sigma_j^2 \right) du \\ & + \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) \\ & + t \int_{\mathbb{R}} \left( \sum_j p_j h_j(x) - h_k(x) + \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \right) \nu(dx) \\ & + \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \tilde{N}(dx, du). \end{aligned} \tag{19}$$

For the integral  $\int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \tilde{N}(dx, du)$  notice that the integrand is not dependent on the time variable. Hence  $\int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) N(dx, du)$  is a compound Poisson process. Since  $\nu(\mathbb{R}) < \infty$ , Theorem 36.5 in Sato [18] tells us that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) N(dx, du) = \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \nu(dx) \quad \text{a.s.}$$

Therefore

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \tilde{N}(dx, du) = 0 \quad \text{a.s.}$$

Now for  $\tilde{\sigma} := \left[ (1 - p_k)^2 \sigma_k^2 + \sum_{j \neq k} p_j^2 \sigma_j^2 \right]^{1/2}$ , we see that  $\tilde{W}(t) := \left[ \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) \right] / \tilde{\sigma}$  is a standard Wiener process, and that  $\tilde{\sigma} \leq 3\sigma_{\max}$ . Thus,  $P_{\mathbf{y}}$  almost surely

$$\begin{aligned} G(t) &\leq G(0) + (K_1 - K_2)t + \tilde{\sigma} \tilde{W}(t) + \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \tilde{N}(dx, du) \\ &\quad + t \int_{\mathbb{R}} \left( \sum_j p_j h_j(x) - h_k(x) + \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \right) \nu(dx). \end{aligned} \quad (20)$$

Therefore, applying the Law of the Iterated Logarithm, we see that

$$\begin{aligned} &\limsup_{t \rightarrow \infty} s_k(t) \exp \left[ t \int_{\mathbb{R}} \left( h_k(x) - \sum_j p_j h_j(x) \right) \nu(dx) - (K_1 - K_2)t - 3\sigma_{\max} \sqrt{t \log \log t} \right] \\ &\leq \limsup_{t \rightarrow \infty} \exp \left[ G(t) + t \int_{\mathbb{R}} \left( h_k(x) - \sum_j p_j h_j(x) \right) \nu(dx) - (K_1 - K_2)t - 3\sigma_{\max} \sqrt{t \log \log t} \right] \\ &\leq \limsup_{t \rightarrow \infty} \exp \left[ G(0) + \int_0^t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \tilde{N}(dx, du) + t \int_{\mathbb{R}} \log \left( \frac{1 + h_k(x)}{1 + \sum_j p_j h_j(x)} \right) \nu(dx) \right. \\ &\quad \left. + \tilde{\sigma} \tilde{W}(t) - 3\sigma_{\max} \sqrt{t \log \log t} \right] = 0. \end{aligned} \quad (21)$$

□

**Acknowledgement.** *The author would like to thank Professors Bob Muncaster, Renming Song, and Lee DeVille for numerous helpful discussions and tremendous guidance.*

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